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NON-LINEAR VIBRATION ANALYSIS OF A TRAVELLING STRING WITH TIME-DEPENDENT LENGTH BY NEW HYBRID LAPLACE TRANSFORM/FINITE ELEMENT METHOD

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A new hybrid Laplace transform/finite element method is proposed to solve the equations governing the vibrations of a string with time-varying length and a weight attached at one end. A set of non-linear gyroscopic and time-varying differential equations is derived by the variable-domain finite element formulation. In this paper, the new hybrid method is extended to solve the variable-domain problems. Proper similarity transform techniques are used to decouple the Laplace transformed equations and make the solutions of the inverse Laplace transform easier. The non-linear terms are linearized by means of Taylor's series expansion. It is found that the simulation results of this new hybrid method converge to the exact solutions and saves a considerable amount of computation time.

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1. INTRODUCTION

The numerical solution of vibration problems of axially moving materials, such as strings and beams, have been described in the literature. This type of problem with time-varying length needs special treatment. Fung and Cheng [1] studied the free vibrations of a nonlinear coupled string/slider system. A string with linear damping, time-dependent length and a weight attached at one end was presented in a series of papers by Kotera *et al*. [2–6]. Yuh and Young [7] investigated the dynamic modelling of an axially moving beam in rotation. Stylianou and Tabarrok [8] investigated the numerical solutions of an axially moving beam by the finite element method. Wang and Wei [9] studied the vibrations of a moving flexible robot beam by the Galerkin method. Kane *et al*. [10], analyzed the problem of a cantilever beam attached to a moving base.

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In previous studies, the hybrid Laplace transform/finite difference method [11, 12] and Laplace transform/finite element method [13, 14] were applied to solve mainly for transient heat conduction problems. The new hybrid Laplace transform/finite element method was also applied to solve two-dimensional and three-dimensional transient heat conduction problems [15, 16]. The non-linear terms were linearized by Taylor's series expansion [17]. Recently, the quasistatic and dynamic responses of a linear viscoelastic beam are solved numerically by using the hybrid Laplace transform/finite element method [18]. The main difficulty in applying the hybrid method is to solve a set linear equation with complex number coefficient and then to take the inverse Laplace transform to the physical plane. The numerical inversion form of the Laplace transform can be written as trigonometric integrals by using the Durbin [19] method. However, in order to avoid the numerically induced oscillations in the solutions, the inverse Laplace transform method of Honig and Hirdes [20], 60 or 100 terms of Fourier series expansion must be taken. Therefore it takes excessive computer time for solutions. The new method is also applied to solve problems such as linear or non-linear irregular geometry of transient heat conduction, viscoelastic behavior of structures and vibration systems.

In this paper the new hybrid Laplace/finite element method [15, 16] is extended to solve variable-domain problems. A proper transform method [21] is applied to the gyroscopic system in order to simplify the solution of inverse Laplace transform. Finite element formulation is usually applied for a fixed size domain. In order to analyze the string with a time-dependent length, a modified finite element technique [8] is used in this study. The main approach of the variable domain finite element formulation is that the number of elements remains fixed, while the element size changes with time. A set of non-linear gyroscopic ordinary differential equations governing the vibrations of a string with the time-varying length is derived. This set of time-varying equations is then solved numerically by the new hybrid Laplace transform/finite element method.

2. FORMULATION OF EQUATION OF MOTION

In this section, Lagrange's equations are employed to derive the governing equations of the travelling string system shown in Figure 1. A plane Cartesian co-ordinate system Oxy is adopted. The axially moving string, neglecting axial deformation, has a mass per unit length ρ and transport velocity $\dot{x}$ and acceleration $\ddot{x}$. The length of the string at time t is specified as $l(t)$. A mass is attached to the lowest end of the string and moves along the (vertical) x direction only. A list of nomenclature is given in Appendix 2.

2.1. KINETIC ENERGY AND STRAIN ENERGY

The transverse displacement of the string at time t and axial position x is described by the field variable $w(x, t)$. The position vector of a point on the deformed string is

$$\mathbf{r} = x(t)\mathbf{i} + w(x, t)\mathbf{j}, \tag{1}$$

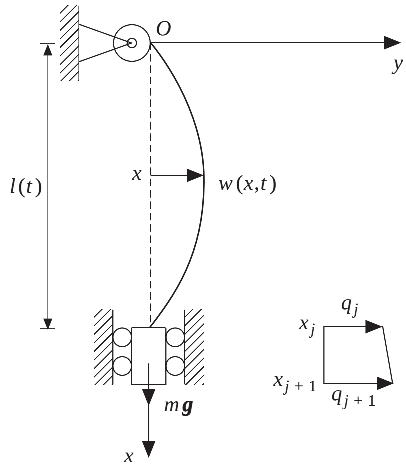

Figure 1. Model of an axially moving string system.

and the corresponding velocity vector is given as

$$\mathbf{v} = \mathrm{d}\mathbf{r}/\mathrm{d}t = \dot{x}\mathbf{i} + (w_t + \dot{x}w_x)\mathbf{j}. \tag{2}$$

The Lagrangian L of the string is

$$L = T - V, \tag{3}$$

where T is its kinetic energy and is expressed as

$$T = \tfrac{1}{2}\int_0^{l(t)} \rho(\dot{x}^2 + w_t^2 + 2w_t w_x \dot{x} + w_x^2 \dot{x}^2)\,\mathrm{d}x, \tag{4}$$

and V is its strain energy and is expressed as

$$V = \tfrac{1}{2}\int_0^{l(t)} [H(x)e + EAe^2]\,\mathrm{d}x, \tag{5}$$

where $e = \frac{1}{2}w_x^2$ represents the engineering strain, E is Young's modulus, A is the cross-section area and H is the tension of the string due to gravity. Since the string is acted upon not only by the weight of the concentrated mass at the lowest end but also its own weight, the tension H is expressed as

$$H(x) = m\mathbf{g} + \rho\mathbf{g}(l - x). \tag{6}$$

The first term in the strain energy integral (5) is due to the tension force H and the second term is due to the non-linear geometric deformation measured from the initially tensioned configuration.

2.2. NEW HYBRID FINITE ELEMENT AND LAPLACE TRANSFORM METHOD

In this section, the finite element model for the transverse vibrations of the string is derived. The current length $l(t)$ of the string is discretized into n elements and each element has equal length. The solution $w(x, t)$ within an element j is approximated by

$$w(x, t) = \mathbf{N}_j(x, l(t))\mathbf{q}_j(t), \qquad x_j \leqslant x \leqslant x_{j+1}, \qquad j = 1, 2, \ldots, n, \tag{7}$$

where the matrix of the shape function $\mathbf{N}_j$ for the linear element is

$$\mathbf{N}_j = \left[\frac{x_{j+1} - x}{x_{j+1} - x_j}, \frac{x - x_j}{x_{j+1} - x_j}\right] = \left[j - \frac{nx}{l}, \frac{nx}{l} - j + 1\right], \tag{8}$$

and the nodal vector $\mathbf{q}_j$ is a function of time

$$\mathbf{q}_j(t) = [q_j(t), q_{j+1}(t)]^{\mathrm{T}}. \tag{9}$$

Due to varying length of the string, $\mathbf{N}_j$ is also time-dependent. Therefore one has

$$w_x = \mathbf{N}_x \mathbf{q}_j, \qquad w_t = \mathbf{N}_t \mathbf{q}_j + \mathbf{N}\dot{\mathbf{q}}_j. \tag{10a, b}$$

Some of derivatives of the shape functions are shown as

$$\mathbf{N}_x = (n/l)[-1, 1], \qquad \mathbf{N}_{xt} = (n\dot{l}/l^2)[1, -1],$$

$$\mathbf{N}_t = (n\dot{l}/l^2)[x, -x], \qquad \mathbf{N}_{tt} = (l\ddot{l} - 2\dot{l}^2)(n/l^3)[x, -x]. \tag{11a–d}$$

Substituting equations (6–8) into equations (4) and (5), the Lagrangian function of element j is derived as

$$L_j = T_j - V_j = \tfrac{1}{2}d_j + \tfrac{1}{2}\mathbf{q}_j^{\mathrm{T}}\mathbf{K}_{j1}\mathbf{q}_j + \mathbf{q}_j^{\mathrm{T}}\mathbf{K}_{j2}\mathbf{q}_j + \tfrac{1}{2}\mathbf{q}_j^{\mathrm{T}}\mathbf{K}_{j3}\mathbf{q}_j$$

$$+ \mathbf{q}_j^{\mathrm{T}}\mathbf{C}_{j1}\dot{\mathbf{q}}_j + \dot{\mathbf{q}}_j^{\mathrm{T}}\mathbf{C}_{j2}\mathbf{q}_j + \tfrac{1}{2}\dot{\mathbf{q}}_j^{\mathrm{T}}\mathbf{M}_j\dot{\mathbf{q}}_j - S_j(\mathbf{q}_j), \tag{12}$$

where d_j is a time function, $\mathbf{M}_j$, $\mathbf{K}_{j1}$, $\mathbf{K}_{j2}$, $\mathbf{K}_{j3}$, $\mathbf{C}_{j1}$, $\mathbf{C}_{j2}$ are all 2×2 matrices; their detailed expressions can be found in Appendix 1 and $S_j(\mathbf{q}_j)$ is the non-linear term of nodal displacements.

The Lagrange's equations for element j are

$$(\mathrm{d}/\mathrm{d}t)\, \partial L_j/\partial \dot{\mathbf{q}}_j^{\mathrm{T}} - \partial L_j/\partial \mathbf{q}_j^{\mathrm{T}} = 0, \tag{13}$$

which are considered to be unconstrained and independent of other elements. Substituting equation (12) into equation (13), one obtains the element governing equations as

$$\mathbf{M}_j\ddot{\mathbf{q}}_j + \mathbf{C}_j\dot{\mathbf{q}}_j + \mathbf{K}_j\mathbf{q}_j + S_j'(\mathbf{q}_j) = 0, \tag{14}$$

where $\mathbf{K}_j = \dot{\mathbf{C}}_{j1}^{\mathrm{T}} + \dot{\mathbf{C}}_{j2} - \mathbf{K}_{j1} - 2\mathbf{K}_{j2} - \mathbf{K}_{j3}$, $\mathbf{C}_j = \mathbf{C}_{j1}^{\mathrm{T}} + \mathbf{C}_{j2} + \dot{\mathbf{M}}_j - \mathbf{C}_{j1} - \mathbf{C}_{j2}^{\mathrm{T}}$ and the non-linear terms

$$S_j'(\mathbf{q}_j) = \frac{\partial S_j}{\partial \mathbf{q}_j^{\mathrm{T}}} = \tfrac{1}{2}EA \int_{x_j}^{x_{j+1}} \mathbf{N}_x^{\mathrm{T}}\mathbf{N}_x\mathbf{q}_j\mathbf{q}_j^{\mathrm{T}}\mathbf{N}_x^{\mathrm{T}}\mathbf{N}_x\mathbf{q}_j\, \mathrm{d}x$$

$$= \frac{EAn^3}{2l^4}\int_{x_j}^{x_{j+1}} \begin{bmatrix} (q_j - q_{j+1})^3 \\ -(q_j - q_{j+1})^3 \end{bmatrix} \mathrm{d}x. \tag{15}$$

The coefficients of the assembled matrices can be obtained directly by using standard FEM assembly procedures. The assembled global equations are represented as

$$\mathbf{M}\ddot{\mathbf{Q}} + \mathbf{C}\dot{\mathbf{Q}} + \mathbf{K}\mathbf{Q} + \mathbf{N}(\mathbf{Q}) = 0, \tag{16}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the assembled system matrices, $\mathbf{Q}$ is the system nodal displacement vector and $\mathbf{N}(\mathbf{Q})$ represents the non-linear terms. It is noted that $\mathbf{M}$, $\mathbf{K}$ are symmetric matrices and $\mathbf{C}$ is an equivalent gyroscopic matrix. These matrices are all time-varying matrices. Equation (16) is a set of non-linear gyroscopic second order differential equations with time-dependent coefficients.

Since the Laplace transform is a method for solving the linear time-invariant equations, assumptions have to be made in governing equations (16). First the non-linear terms (15) are linearized by a Taylor's series expansion. Second, the terms with time-dependent coefficients are also taken as a Taylor's series expansion about the previous time step. The set of second order non-linear differential equations (16) is then reduced approximately to a set of linear time-invariant differential equations

$$\hat{\mathbf{M}}\ddot{\mathbf{Q}} + \hat{\mathbf{C}}\dot{\mathbf{Q}} + \hat{\mathbf{K}}\mathbf{Q} = \mathbf{Y}, \tag{17}$$

where the terms with a circumflex are the nodal solutions at a previous time step,

$$\mathbf{Y} = -\hat{\mathbf{F}} + \hat{\mathbf{M}}\ddot{\mathbf{Q}} + \hat{\mathbf{C}}\dot{\mathbf{Q}} + \hat{\mathbf{K}}\hat{\mathbf{Q}} - \mathbf{M}\ddot{\mathbf{Q}} - \mathbf{C}\dot{\mathbf{Q}} - \mathbf{K}\mathbf{Q} \text{ and } \hat{\mathbf{F}} = \sum_{j=1}^{n} \hat{\mathbf{F}}_j,$$

$$\hat{\mathbf{F}}_j = \frac{EAn^4}{2l^4}\begin{bmatrix} -2\hat{q}_j^3 + 2\hat{q}_{j+1}^3 + 6\hat{q}_j^2\hat{q}_{j+1} - 6\hat{q}_j\hat{q}_{j+1}^2 \\ 2\hat{q}_j^3 - 2\hat{q}_{j+1}^3 - 6\hat{q}_j^2\hat{q}_{j+1} + 6\hat{q}_j\hat{q}_{j+1}^2 \end{bmatrix}. \tag{18}$$

In the following, two different cases are discussed separately to illustrate the present approaches for solving the string system. Case 1 studies a non-linear time-invariant system with symmetric system matrices while case 2 studies a non-linear gyroscopic time-varying system.

Case 1: [$\mathbf{C}$] *is symmetric matrix*

In the case of a non-linear string with fixed length, the system becomes time-invariant and the gyroscopic term $\mathbf{C}\dot{\mathbf{Q}}$ in equation (16) will vanish. A diagonal damping matrix is added in equation (16) and all the system matrices can be diagonalized simultaneously by using a similarity transform method. For this case $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ and $\mathbf{N}(\mathbf{Q})$ are constants in equation (16). Only the non-linear term $\mathbf{N}(\mathbf{Q})$ is necessary to be considered in the Taylor's series expansion. The linearized equations are

$$\mathbf{M}\ddot{\mathbf{Q}} + \mathbf{C}\dot{\mathbf{Q}} + \mathbf{K}\mathbf{Q} = -\hat{\mathbf{F}}. \tag{19}$$

By taking a Laplace transform of equation (19) one obtains

$$(\mathbf{s}^2\mathbf{M} + \mathbf{s}\mathbf{C} + \mathbf{K})\bar{\mathbf{Q}} = \mathbf{f}, \tag{20}$$

where $\mathbf{f} = (\mathbf{sM} + \mathbf{C})\mathbf{Q}(0) + \mathbf{M}\dot{\mathbf{Q}}(0) - (1/\mathbf{s})\hat{\mathbf{F}}$, $\mathbf{s}$ is a complex number and $\bar{\mathbf{Q}}$ is a $(n \times 1)$ vector representing the Laplace transform of unknown displacement functions.

Since $\mathbf{M}$ is positive definite, all of its eigenvalues are positive. One can define the following transform matrix

$$\mathbf{R} = [\mathbf{e}_1/\sqrt{\mu_1}, \mathbf{e}_2/\sqrt{\mu_2}, \ldots, \mathbf{e}_n/\sqrt{\mu_n}], \tag{21}$$

where $\mathbf{e}_i$ is the ith eigenvector of $\mathbf{M}$ corresponding to its ith eigenvalue μ_i and obtains

$$\mathbf{R}^{\mathrm{T}}\mathbf{M}\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{22}$$

Sequentially, using the following transformation

$$\bar{\mathbf{Q}} = \mathbf{R}\bar{\mathbf{Q}}^{(1)}, \tag{23}$$

and pre-multiplying $\mathbf{R}$, equation (20) becomes

$$(\mathbf{s}^2\mathbf{I} + \mathbf{sT} + \mathbf{G})\bar{\mathbf{Q}}^{(1)} = \mathbf{f}^{(1)}, \tag{24}$$

where $\mathbf{I}$ is the unit matrix, $\mathbf{T} = \mathbf{R}^{\mathrm{T}}\mathbf{C}\mathbf{R}$, $\mathbf{G} = \mathbf{R}^{\mathrm{T}}\mathbf{K}\mathbf{R}$ and $\mathbf{f}^{(1)} = \mathbf{R}^{\mathrm{T}}\mathbf{f}$.

Now, $\mathbf{G}$ is a symmetric matrix, and can be reduced to a diagonal form after setting

$$\bar{\mathbf{Q}}^{(1)} = \mathbf{P}\bar{\mathbf{Q}}^{(2)}, \tag{25}$$

where $\mathbf{P}$ is formed from the eigenvectors of $\mathbf{G}$. Therefore, substituting equation (25) into equation (24) and pre-multiplying equation (24) by $\mathbf{P}^{\mathrm{T}}$ yields

$$\left\{ \mathbf{s}^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \mathbf{s}\mathbf{P}^{\mathrm{T}}\mathbf{T}\mathbf{P} + \begin{bmatrix} \kappa_i & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \kappa_n \end{bmatrix} \right\} \bar{\mathbf{Q}}^{(2)} = \mathbf{P}^{\mathrm{T}}\mathbf{f}^{(1)} = \mathbf{f}^{(2)}, \tag{26}$$

where κ_i is the ith eigenvalue of the matrix $\mathbf{G}$ and $\mathbf{P}^{\mathrm{T}}\mathbf{P} = \mathbf{I}$.

Defining $\mathbf{T}_1 = \mathbf{P}^{\mathrm{T}}\mathbf{T}\mathbf{P}$ and $\bar{\mathbf{Q}}^{(2)} = \mathbf{W}\bar{\mathbf{Q}}^{(3)}$, equation (26) can be further transformed to

$$\left\{ \mathbf{s}^2\mathbf{I} + \mathbf{s} \begin{bmatrix} \beta_1 & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \beta_n \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} \right\} \bar{\mathbf{Q}}^{(3)} = \mathbf{f}^{(3)}, \tag{27}$$

where $\mathbf{W}$ is made up of the eigenvectors of $\mathbf{T}_1$; β_i is the ith eigenvalue of the matrix $\mathbf{T}_1$, $\mathbf{f}^{(3)} = \mathbf{W}^{\mathrm{T}}\mathbf{f}^{(2)}$, $\mathbf{W}^{\mathrm{T}}$ diag $[\kappa_i]\mathbf{W} =$ diag $[\lambda_i]$ and $\mathbf{W}^{\mathrm{T}}\mathbf{W} = \mathbf{I}$.

Assembling equation (27), one gets the decoupled relations

$$\{\bar{q}_i^{(3)}\} = \{f_i^{(3)}/(\lambda_i + \mathbf{s}\beta_i + \mathbf{s}^2)\}, \qquad (i = 1, 2, \ldots, n). \tag{28}$$

Therefore, the inverse Laplace transform of $\bar{q}_i^{(3)}(\mathbf{s})$ can be obtained by partial fractions. The merits of this proposed transformation is to deduce the decoupled form and the inverse Laplace transform solution can be obtained easily. Finally, the displacement $\bar{\mathbf{Q}}$ can be obtained by

$$\bar{\mathbf{Q}} = \mathbf{RPW}\bar{\mathbf{Q}}^{(3)}. \tag{29}$$

Case 2: **C** *is a gyroscopic matrix*, **M**, **C** *and* **K** *are time-varying matrices*

In this case of a non-linear string, the length is time dependent and **C** is a gyroscopic matrix. The similarity transform method used in case 1 cannot be applied to this gyroscopic system. In this formulation, a special transformation matrix function [21] of the general co-ordinates will be used to change the system with the gyroscopic matrix. Equation (17) can be reduced to

$$\ddot{\mathbf{Q}} + 2\mathbf{A}\dot{\mathbf{Q}} + \mathbf{BQ} = \hat{\mathbf{M}}^{-1}\mathbf{Y}, \tag{30}$$

where

$$\mathbf{A} = \tfrac{1}{2}\hat{\mathbf{M}}^{-1}\hat{\mathbf{C}}, \qquad \mathbf{B} = \hat{\mathbf{M}}^{-1}\hat{\mathbf{K}}. \tag{31, 32}$$

Substituting $\mathbf{Q} = \mathbf{V}(t)\mathbf{Q}_1$ into equation (30), one obtains

$$\mathbf{V}(t)\ddot{\mathbf{Q}}_1 + 2(\dot{\mathbf{V}}(t) + \mathbf{AV}(t))\dot{\mathbf{Q}}_1 + (\ddot{\mathbf{V}}(t) + 2\mathbf{A}\dot{\mathbf{V}}(t) + \mathbf{BV}(t))\mathbf{Q}_1 = \hat{\mathbf{M}}^{-1}\mathbf{Y}, \tag{33}$$

where $\mathbf{V}(t)$ makes the coefficient matrix of $\dot{\mathbf{Q}}_1$ zero, i.e., $\dot{\mathbf{V}}(t) + \mathbf{AV}(t) = 0$. The solutions are

$$\mathbf{V} = \exp(-\mathbf{A}t). \tag{34}$$

Using equation (34), equation (33) can be rewritten as

$$\ddot{\mathbf{Q}}_1 + \mathbf{DQ}_1 = \hat{\mathbf{M}}^{-1}\mathbf{Y}^{(1)}, \tag{35}$$

where $\mathbf{D} = \exp(\mathbf{A}t)(\mathbf{B} - \mathbf{A}^2)\exp(-\mathbf{A}t)$ and $\mathbf{Y}^{(1)} = \hat{\mathbf{M}}\exp(\mathbf{A}t)\hat{\mathbf{M}}^{-1}\mathbf{Y}$.

The necessary and sufficient condition for **D** to be a constant matrix is that its differential matrix with respect to time is zero. In this situation, $\mathbf{AB} = \mathbf{BA}$ must be satisfied and then one has $\mathbf{D} = \mathbf{B} - \mathbf{A}^2$. Substituting equations (31) and (32) into equation (35) leads to the following equivalent system

$$\hat{\mathbf{M}}\ddot{\mathbf{Q}}_1 + \hat{\mathbf{K}}_{eq}\mathbf{Q}_1 = \mathbf{Y}^{(1)}, \tag{36}$$

where $\hat{\mathbf{K}}_{eq} = \hat{\mathbf{K}} - \frac{1}{4}\hat{\mathbf{C}}\hat{\mathbf{M}}^{-1}\hat{\mathbf{C}}$. It is shown in reference [21] that if $\langle\hat{\mathbf{M}}, \hat{\mathbf{K}}\rangle$ of system (17) gives no repeated eigenvalues, then equivalent system (36) is valid. This is the case for the present travelling string system. From equation (36), one can easily take the inverse Laplace transform after the similarity transform method has been applied to decouple the system. Taking the Laplace transform of equation (36), one has

$$\mathbf{s}^2\hat{\mathbf{M}}\bar{\mathbf{Q}}_1 + \hat{\mathbf{K}}_{eq}\bar{\mathbf{Q}}_1 = \bar{\mathbf{Y}}^{(1)} + \hat{\mathbf{M}}(\mathbf{s}\{q_1(0)\} + \{\dot{q}_1(0)\}). \tag{37}$$

In the following, a similarity transform method is used on the matrices, i.e., $\bar{\mathbf{Q}}_1 = \mathbf{R}_1^{(1)}\bar{\mathbf{Q}}_1^{(1)}$ and $\bar{\mathbf{Q}}_1^{(1)} = \mathbf{P}_1\bar{\mathbf{Q}}_1^{(2)}$, then one has

$$\bar{\mathbf{Q}}_1^{(2)} = \frac{1}{s^2+\gamma_i}\mathbf{P}_1^{-1}\mathbf{R}_1^{(1)\mathrm{T}}(\hat{\mathbf{M}}(\mathbf{s}\{q_1(0)\} + \{\dot{q}_1(0)\})) + \frac{1}{s^2+\gamma_i}\mathbf{P}_1^{-1}\mathbf{R}_1^{(1)\mathrm{T}}\bar{\mathbf{Y}}^{(1)}, \quad (38a)$$

where $\mathbf{R}_1^{(1)}$ are the eigenvectors of $\hat{\mathbf{M}}$. $\mathbf{P}_1$, γ_i are the modal matrices and the ith eigenvalue of the matrix $\mathbf{R}_1^{(1)\mathrm{T}}\hat{\mathbf{K}}_{eq}\mathbf{R}_1^{(1)}$. $\hat{\mathbf{M}}$ and $\hat{\mathbf{K}}_{eq}$ are the nodal solutions at a previous time step.

Therefore, the inverse Laplace transform of $\bar{\mathbf{Q}}_1^{(2)}$ can be obtained by partial fractions expansion. Finally, one obtains the displacements

$$\bar{\mathbf{Q}}_1 = \mathbf{R}_1^{(1)}\mathbf{P}_1\bar{\mathbf{Q}}_1^{(2)}, \qquad \mathbf{Q} = \exp(-\mathbf{A}t)\mathbf{Q}_1. \quad (38b, c)$$

3. NUMERICAL RESULTS AND DISCUSSION

This paper is concerned with the non-linear vibrations of the string with time-varying length and a weight attached at one end. By Hamilton's principle, the governing equation is given from equations (4) and (5) as

$$\rho\omega_{tt} + 2\rho\dot{x}w_{xt} + \rho\ddot{x}w_x + (\rho\dot{x}^2 - H)w_{xx} - \tfrac{3}{2}EAw_x^2w_{xx} = 0, \qquad 0 \leqslant x \leqslant l, \quad (39)$$

and the boundary conditions are

$$w(0, t) = 0, \qquad w(l, t) = 0. \quad (40, 41)$$

In the following sections, three examples are analyzed by the new hybrid Laplace transform/finite element method and the results are compared with those obtained by the Runge–Kutta method. All the work was done on a 166 MHz personal computer, the desired accuracy of 10^{-9} is specified in MATLAB and the integration time step is 0·04 s for the Runge–Kutta method.

3.1. EXAMPLE 1 (FIXED–FIXED LINEAR CASE)

The simplest case is the string fixed at both sides. The geometrically non-linear term $\frac{3}{2}EAw_x^2w_{xx}$ and the travelling velocity $\dot{x}$ and acceleration $\ddot{x}$ of equation (39) are not considered in this case. The system is a standard linear string equation and is represented as follows:

$$\partial^2 w/\partial t^2 = \alpha^2\frac{\partial^2 w}{\partial x^2}, \qquad 0 \leqslant x \leqslant l \qquad t > 0; \quad (42)$$

the boundary conditions are

$$w(0, t) = 0, \qquad w(l, t) = 0, \quad (43a, b)$$

where $\alpha = \sqrt{T_0/\rho}$ and T_0 is the initial constant tension. If the initial condition is

$$w(x, 0) = 0{\cdot}01\sin(\pi x/l), \quad (44)$$

the analytical solution for this problem is

$$w(x,\, t) = \sum_{n=1}^{\infty} \left\{ A_n \cos \frac{\alpha n \pi}{l}\, t \right\} \sin\left(\frac{n\pi}{l}\, x\right), \tag{45}$$

where

$$A_n = \frac{2}{l} \int_0^l 0{\cdot}01 \sin(\pi x) \sin\left(\frac{n\pi}{l}\, x\right) \mathrm{d}x.$$

The solutions obtained by using new Laplace transform/finite element method with different element numbers $n = 8$, 16 and 32 are compared with those analytic solutions (45) and shown in Figure 2. It is seen that the solutions of the nodal displacements by the present new method with the element numbers $n = 16$, 32 are almost the same as the exact solutions. It is found by simulation that this new hybrid method is stable and convergent to the exact solution. The simulation cost time will increase naturally for increasing element number. However, they are almost the same for different transient vibrations $t = 0{\cdot}04$, $0{\cdot}5$ and 1 s, because the Laplace transform with no time step can obtain a fast determination of the solutions for linear time-invariant system.

3.2. EXAMPLE 2 (FIXED–FIXED NONLINEAR CASE)

In this example, the non-linear term $\frac{3}{2}EAw_x^2 w_{xx}$ is considered and its governing equation is given as

$$\partial^2 w/\partial x^2 = (1/\alpha^2)\, \partial^2 w/\partial t^2 - \tfrac{3}{2} EA w_x^2 w_{xx}; \tag{46}$$

the boundary and initial conditions are the same as those in equation (43a, b) and (44).

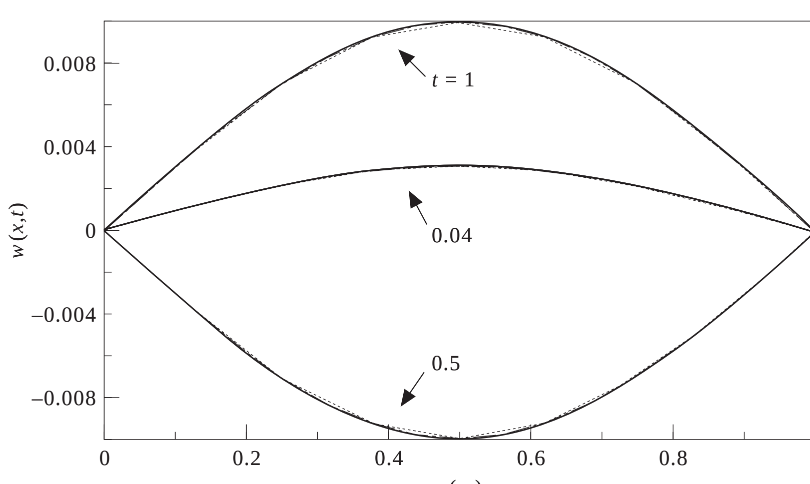

Figure 2. Comparison of the solutions of nodal displacements for the linear fixed–fixed string with cost times 0·06 s ($n = 8$); 0·33 s ($n = 16$); 1·32 s ($n = 32$). Key: . . . . , $n = 8$; –·–·–, $n = 16$; –––, $n = 32$; ——, exact.

Since the system is non-linear, there is no exact solution for this case. The results of simulation by the Runge–Kutta method and the new proposed method are compared in Figure 3. It is seen that the string has an amplitude near 0·01 (m) at $t = 5$ s. By using the new method, the cost times of computer simulation are 0·55, 0·67 and 2·31 s for the element numbers $n = 8$, 16, 32, respectively. But the simulation time is 232·7 s for the Runge–Kutta method.

3.3. EXAMPLE 3 (THE NON-LINEAR STRING WITH TIME-VARYING LENGTH)

The non-linear vibrations of the string with time-varying length are investigated in this example. The following numerical simulations will show that nodal displacements by the new method are almost identical with those by the Runge–Kutta method, but this new method will save a lot of computation time.

3.3.1. *Sinusoidal axial motion*

The axial velocity is given as

$$v(t) = v_0 + v_1 \sin \omega t, \tag{47}$$

where v_0 is the steady state velocity, and v_1 and ω are amplitude and frequency of the axial perturbation velocity respectively. Then, the length and the axial acceleration can be respectively expressed as

$$l(t) = l_0 + v_0 t - (v_1/\omega)\cos \omega t, \qquad a(t) = \omega v_1 \cos \omega t. \tag{48, 49}$$

In this case, free vibrations are excited by a distributed initial displacement of a half-sine wave and released from rest. The initial displacement and velocity are respectively

$$w(x, 0) = w_0 \sin (x\pi/l_0), \qquad w_t(x, 0) = 0, \tag{50a, b}$$

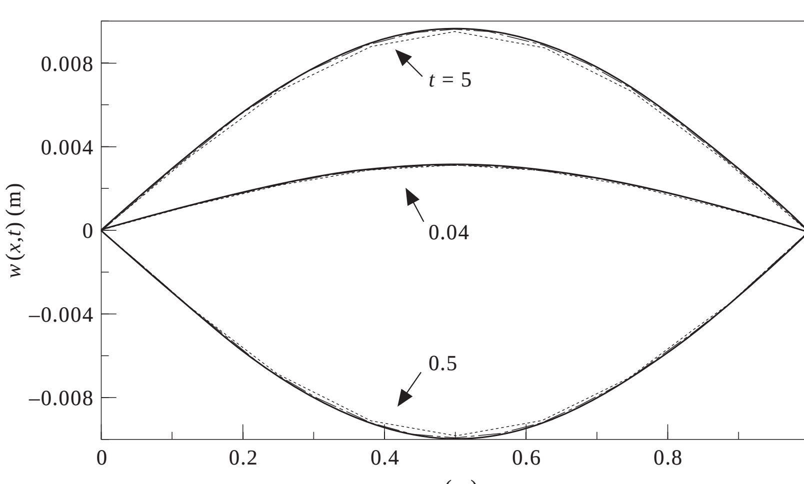

Figure 3. Comparison of the solutions of nodal displacements for the non-linear fixed–fixed string with cost times for upper curve 0·55 s (8); 0·67 s (16); 2·31 s (32); middle curve 0·10 s (8); 0·22 s (16); 0·38 s (32) and bottom curve 0·33 s (8); 0·45 s (16); 0·61 s (32). Key as for Figure 2 except for ——, Runge–Kutta.

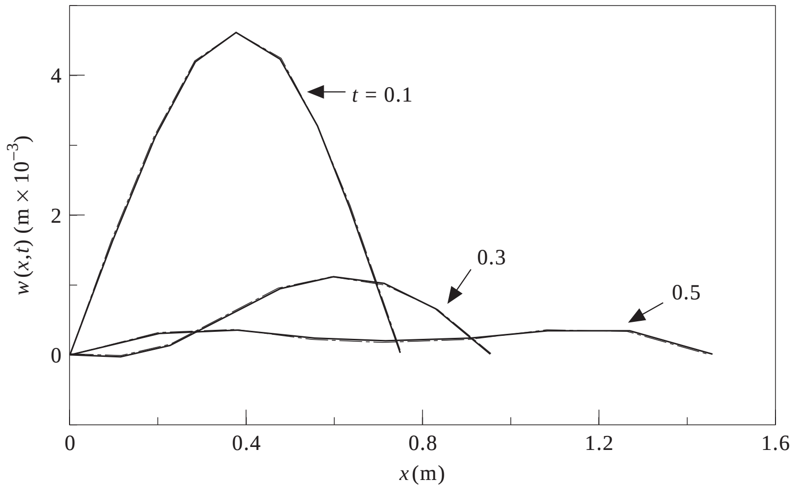

Figure 4. Comparison of the solutions of nodal displacements for the non-linear system with the sinusoidal axial-motion, $v = v_0 + v_1 \sin \omega t$ ($v_0 = 1$ m/s, $v_1 = 3$ m/s). Key: ——, Runge–Kutta method; –·–·–, new hybrid Laplace transform.

where $w_0 = 0{\cdot}005$ m is the initial amplitude. Parameters $m\mathbf{g} = 100$ N, $v_0 = 1$ m/s, $v_1 = 3$ m/s, $\omega = 62{\cdot}83$ rad/s, $\rho = 1$ kg/m, $EA = 1000$ N and $l_0 = 0{\cdot}5$ m are used in simulations. Three shapes at $t = 0{\cdot}1$, 0·3, 0·5 s with $n = 8$ are shown in Figure 4. The computation times for the whole solutions by using the new method (–·–·–), are respectively 0·27, 0·72, 1·09 s and by the Runge–Kutta method (——) are 53·50, 160·50, 267·51 s. The nodal displacements by using the new hybrid Laplace transform are almost the same as those obtained by using the Runge–Kutta method with integration time step 0·04 s and desired accuracy 10^{-9}.

3.3.2. *Parabolic axial motion*

The following time-dependent length $l(t)$ is used to generate this type of axial motion

$$l(t) = l_0 + v_0 t + \tfrac{1}{2} a_0 t^2, \tag{51}$$

where l_0, v_0 and a_0 are the initial length, velocity and acceleration of the string respectively. Parameters $l_0 = 0{\cdot}5$ m, $v_0 = 0$, $a_0 = 3$ m/s^2, $EA = 1000$ N and $\rho = 1$ kg/m are used in simulations. Three shapes at $t = 0{\cdot}15$, 0·20, 0·50 s with $n = 8$ are shown in Figure 5. The computation times for the whole solutions by use of the new method (–·–·–) are 0·3, 0·6, 1·09 (sec) and by the Runge–Kutta method (——) 80·25, 107, 267·50 s.

The transient vibrations for the longer times with $n = 8$ are shown in Figure 6. The solutions of nodal displacements obtained are compared by the new method (–·–·–) and the Runge–Kutta method (——). Figure 6(a) is for sinusoidal axial motion with $l_0 = 0{\cdot}5$ m, $v_0 = 1$ m/s, $v_1 = 3$ m/s and $t = 1$, 2, 2·5 s while Figure 6(b) is for parabolic axial motion with $l_0 = 0{\cdot}5$ m, $v_0 = 1$ m/s, $a_0 = 3$ m/s^2 and $t = 1$, 2, 3 s. The computation times for the whole solution are 2·03, 3·91, 4·99 s for the new method, and 535, 1070, 1337·5 s for the Runge–Kutta method in Figure 6(a) and 2·03, 3·90, 5·66 s and 535, 1070, 1605 respectively in Figure 6(b).

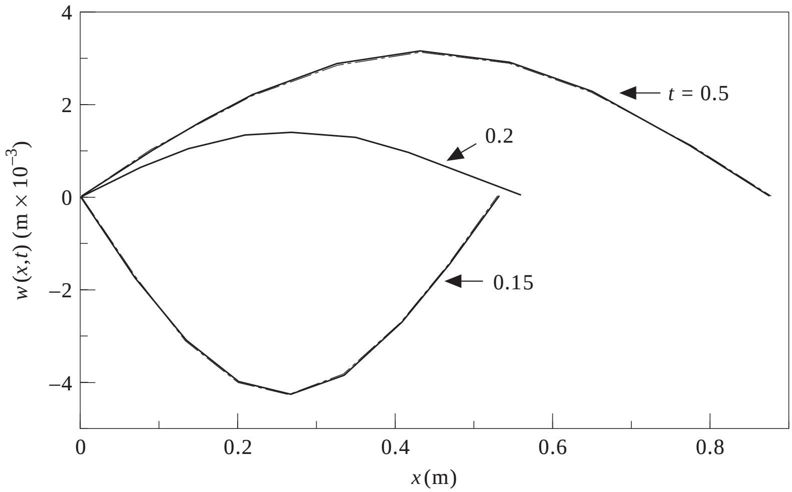

Figure 5. Comparison of the solutions of nodal displacements for the non-linear system with the parabolic axial motion ($a_0 = 3$ m/s^2, $v_0 = 0$ m/s, $l_0 = 0{\cdot}5$ m). Key as for Figure 4.

The new hybrid method also saves much computation time in the sinusoidal and parabolic axial motions of a non-linear travelling string.

4. CONCLUSION

The non-linear vibrations of a string with time-varying length and a weight attached at one end have been successfully solved by the new hybrid Laplace transform/finite element method which can save a lot of computation time. Using the transform method on the matrix of the complex number coefficients, this hybrid method can be used to handle the non-linear gyroscopic problems with many nodes. It is seen that Laplace transform with no time step produces a fast determination of the solution for a linear time-invariant system.

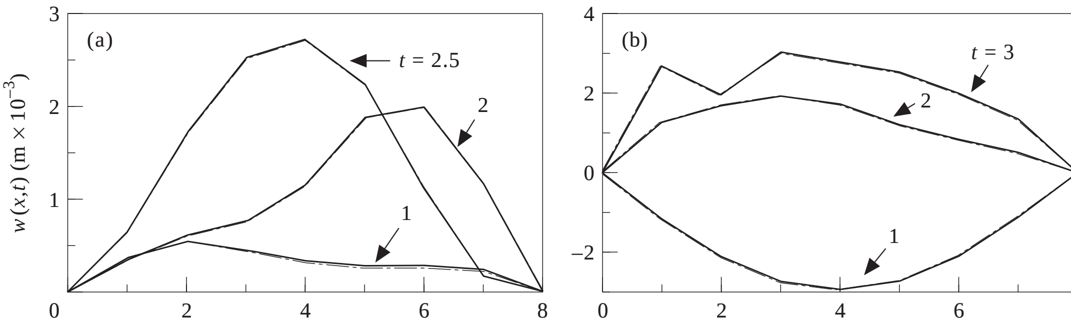

Figure 6. Comparison of the solutions of nodal displacements for the long time transient vibrations. (a) Sinusoidal axial-motion and $t = 1$, 2, 2·5 s, (b) parabolic axial-motion and $t = 1$, 2, 3 s. Key: –·–·–, the new method; ——, Runge–Kutta method.

According to the present work, conclusions are drawn as follows:

(1) The new hybrid Laplace transform method combined with a similarity transform method can speed up the solutions.

(2) The terms containing the axial velocity in the governing equation are included in a gyroscopic matrix from a finite element formulation. The gyroscopic and stiffness matrices cannot be made diagonal simultaneously from the traditional similarity transformation. A special transformation is used to decouple the Laplace transformed equations and easily obtain the inverse Laplace transform.

(3) The non-linear time-varying system can be solved by the proposed method. The benefit of the new method is especially suitable for the long-time integration for the non-linear gyroscopic time-varying systems.

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APPENDIX 1

$$d_j = \int_{x_j}^{x_{j+1}} \rho \dot{x}^2 \, \mathrm{d}x, \qquad \mathbf{M}_j = \frac{\rho l}{6n}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{K}_{j1} = \frac{\rho \dot{l}^2}{3nl}(3j^2 - 3j + 1)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

$$\mathbf{K}_{j2} = \frac{\rho \dot{l}^2}{2l}(-2j + 1)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{K}_{j3} = \frac{n}{l}(\rho \dot{l}^2 - m\mathbf{g})\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

$$\mathbf{C}_{j1} = \frac{\rho \dot{l}}{2l}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \qquad \mathbf{C}_{j2} = \frac{\rho \dot{l}}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix},$$

$$S_j(\mathbf{q}_j) = \tfrac{1}{8} EA \int_{x_j}^{x_{j+1}} \mathbf{q}_j^{\mathrm{T}} \mathbf{N}_x^{\mathrm{T}} \mathbf{N}_x \mathbf{q}_j \mathbf{q}_j^{\mathrm{T}} \mathbf{N}_x^{\mathrm{T}} \mathbf{N}_x \mathbf{q}_j \, \mathrm{d}x.$$

APPENDIX 2: NOMENCLATURE

A	cross-section area of string
a	acceleration of the string
E	Young's modulus of string
H	initial tension of the string
L	Langrangian function
l_0	initial length of the string
$l(t)$	length of the string at time t
$m\mathbf{g}$	weight of the concentrated mass
$\mathbf{N}_j$	shape function
n	total number of finite elements in the string
$\mathbf{q}_j$	nodal variable vector
$\mathbf{r}$	position vector of a point on the string
S_j	non-linear term of the nodal displacements
$\mathbf{s}$	complex number
T	kinetic energy of the string
V	strain energy of the string
v	velocity of the string
v_0	steady-state velocity
v_1	amplitude of the axial perturbative velocity
$w(x, t)$	transverse displacement of the string
x	position in axial direction
$\dot{x}$	velocity of axial motion
$\ddot{x}$	acceleration of axial motion

Greek symbols

ρ	mass density per unit length
ω	radial frequency

Superscript

$-$	Laplace transform
T	transposed matrix